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NON-UNIFORM FLOW IN MULTISTAGE AXIAL COMPRESSORS

Frank E. Marble

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angle, including stall, and whether the profiles may be characterized by a sufficiently small number of parameters to make the technique attractive. The present paper examines a specific example of distorted inlet flow through the two-dimensional annulus of a multistage compressor which can be solved completely. It is shown that the essential features of this exact solution, including stall, may be described by a two-parameter family of profiles and that an integral technique, utilizing these elementary profiles, will yield essentially the same results. While it is not clear that comparable success would hold for the three-diminsional problem, the results confirm the contention that the two-dimensional problem may be treated with acceptable accuracy by an integral technique.

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ABSTRACT

It has been suggested by the author that some aspects of severely distorted flow into multistage compressors may be examined utilizing an integral technique. The general idea of the proposed technique is clear enough; the appropriate equations of motion and energy are integrated peripherally and radially, using reasonable assumptions for the distributions of velocity and thermodynamic properties, and thereby reduced to ordinary non-linear differential equations for the parameters that describe the distributions. The questions that arise are whether the cascade characteristics may be described appropriately over wide variations of inlet angle, including stall, and whether the profiles may be characterized by a sufficiently small number of parameters to make the technique attractive.

The present paper examines a specific example of distorted inlet flow through the two-dimensional annulus of a multistage compressor which can be solved completely. It is shown that the essential features of this exact solution, including stall, may be described by a two-parameter family of profiles and that an integral technique, utilizing these elementary profiles, will yield essentially the same results. While it is not clear that comparable success would hold for the three-dimensional problem, the results confirm the contention that the two-dimensional problem may be treated with acceptable accuracy by an integral technique.

1. INTRODUCTION

Non-uniform flow conditions entering multistage compressors are among the most significant factors that limit the performance of these units and restrict the operation of engines of which they are components.[1, 2] Under some circumstances the non-uniformities are attenuated in the first few stages while otherwise they may persist through the entire machine, causing severe performance degradation and unacceptable blade loads. The development or smoothing out of severely distorted inlet states is a complex cooperative phenomenon among many successive stages[3, 4, 5] and is not readily understood in terms of individual blade row performance. It is the aim of analytical work in this field to formulate such prolems within the requisite accuracy and to find methods by which physically relevant results may be extracted. It is the contention of some workers, including the author, that, preferably, such formulations and methods should lead relatively simply from a physical model to significant results.

One contender for the appropriate analytical technique is an integral method in which the equations of motion and energy are integrated over the compressor annulus at a fixed point along the compressor axis. The most familiar example of this technique is the Karman integral method for the boundary layer, although the general idea has found wide application. The applicability of the technique to the problem of asymmetric compressor flow rests not only upon whether it is simpler than a detailed numerical calculation, but even more critically, whether the velocity, pressures and temperature profiles may be described adequately by a small number of parameters and whether the blade characteristic can be specified reasonably over a wide range in angle of attack. One can not settle these questions for all circumstances, but one example in which an "exact" calculation and an integral method are in agreement would be a guide to the merits and limitations of the method.

It is the aim of this paper to construct such an example in which the exact solution can be carried out. The problem utilizes a blade row and stalling characteristic and a solution technique introduced by the author(6) several years ago in the analysis of propagating stall in single blade rows. To make matters simpler, the example considers a two-dimensional annular section of the compressor, represented as a repetitive flow pattern in the plane, and the fluid field is regarded as uncompressible and inviscid although the individual blade rows impose losses on the flow across them. The essential questions to be asked of the example is whether the profiles-especially the velocity profile-can be described in terms of a small number of parameters. If the required number is few, then the possibility exists that the integral approach may prove useful in treating problems of distorted inlet flow.

2. BLADE CHARACTERISTICS AND REPRESENTATION OF STALL

Consider the flow of an incompressible fluid through a two-dimensional cascade utilizing the notation of Figure 1; β_1 and β_2 are the relative inflow and discharge angles under conditions of steady, uniform flow and U is the constant axial velocity. We replace the cascade by an actuator surface coincident with the y axis. Under these circumstances the pressures ahead and behind the cascade are uniform and denoted p_1 and p_2 respectively. The characteristics of the cascade will be described by the turning angle, $\beta_2 - \beta_1$, and the static pressure rise, $p_2 - p_1 \equiv \Delta p$, in terms of the inlet angle β_1. We shall take these characteristics to be those given in Figures 2 and 3, a form introduced in Ref. 6, for the analysis of propagating stall. The pressure rise, Δp, is a discontinuous function of inlet angle and drops to zero at the local inlet stalling angle, β_1^*, but is continuous and smooth for angles $\beta_1 < \beta_1^*$ The outlet angle, β_2, is a smooth function of β_1 through the stall. The stall is thus characterized by the absence of static pressure rise in the separated blade channel, that is, the relative diffusion drops to zero.

Now we consider non-steady flows in the cascade to be periodic over a range $2\pi R$, corresponding to the periodicity about the compressor annulus, and to be reducible to steady flow by an appropriate velocity of translation along the y-axis. A disturbance that enters from a fixed obstruction ahead of the inlet is steady with respect to the stationary blade rows, but we must move with a vertical velocity ΩR with respect to a rotor blade row to achieve a steady flow field. In the case of self-induced disturbances, such as a stall propogation that moves with an angular velocity ω, we must move with a velocity $-\omega R$ along a stator and with a velocity $\Omega R - \omega R$ along a rotor in order to achieve a steady flow field. This is possible because the actuator surface concept suppresses effects of individual blades and these are the only identifiable features of blade row geometry.

Where the appropriate transformation along the y-axis has been made, we shall call the angles of the steady flow, θ_1 and θ_2 upstream and downstream respectively; the relations between the relative flow angles β_1, β_2 and the steady absolute angles θ_1, θ_2 are

$$\tan\theta_1 - \tan\beta_1 = \frac{\omega R}{U} \qquad 1$$

for a stator and

$$\tan\theta_1 - \tan\beta_1 = \frac{\omega R}{U} - \frac{\Omega R}{U} \qquad 2$$

for a rotor where ω is the absolute angular velocity of the disturbance and Ω is the angular velocity of a rotor. For a stationary disturbance introduced at the inlet, we set $\omega = 0$; for a propagating stall, we determine ω as a characteristic value.

3. NON-UNIFORM FLOW IN A SINGLE BLADE ROW

Under any of the circumstances just considered, the flow field induced by a single blade row may be described as in Figure 4, in which any existing stalled region of length $(2\pi R)\alpha$ is located symmetrically with respect to the x-axis. The flow angles θ_1 and θ_2 and the tangential velocities V_1 and V_2 are those that exist in the absence of stall. We are concerned with small deviations from these two uniform fields introduced by a stall or an upstream distortion.

The perturbation equations applicable to either upstream or downstream flow fields, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 3$$

$$U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad 4$$

$$U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \qquad 5$$

The disturbance ϑ of the flow angle from the steady flow angle may be expressed in terms of the velocity perturbations and if we retain only terms linear in perturbation quantities, this relation is

$$(1+\tan^2\theta)\vartheta = \frac{v}{U} - \frac{u}{U}\tan\theta \qquad 6$$

Now if we utilize equation 3 to eliminate the term $\frac{\partial u}{\partial x}$ in favor of $\frac{\partial v}{\partial y}$ in the x-component of momentum conservation, equation 4, we obtain $-U^2\frac{\partial}{\partial y}\left(\frac{v}{U} - \frac{u}{U}\tan\theta\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$ and, upon introducing the flow angle perturbation, equation 6,

$$(1+\tan^2\theta)U^2\frac{\partial\vartheta}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad 7$$

Similar operations on the second equation of motion, equation 5, yields

$$(1+\tan^2\theta)U^2\frac{\partial\vartheta}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad 8$$

If, therefore, we choose two new dependent variables

$$\Theta = \vartheta \tag{9}$$

$$P = \frac{p}{\rho(U^2+V^2)} \equiv \frac{p}{\rho U^2(1+\tan^2\Theta)} \tag{10}$$

the two equations of motion, in the forms of equations 7 and 8, become

$$\frac{\partial P}{\partial x} = \frac{\partial \Theta}{\partial y} \tag{11}$$

$$\frac{\partial P}{\partial y} = -\frac{\partial \Theta}{\partial x} \tag{12}$$

Thus the complex variable $P + i\Theta$ may be considered an analytic function of the complex variable $x + iy = z$

$$P + i\Theta = W(z) \tag{13}$$

This description of the pressure and flow angle perturbations in terms of an analytic function does not, of course, necessitate that the flow be irrotational. The irrotational part of the field is determined by $W(z)$, but additional "shear flow" solutions may be superimposed because, when the flow is steady, such flows carry no pressure field and induce no angle variations. In the present problem the admissible shear flows are easily obtained by writing a linear combination of equations 4 and 5 to give

$$\left\{\cos\Theta\frac{\partial}{\partial x} + \sin\Theta\frac{\partial}{\partial y}\right\}\left\{\frac{u}{U} + \frac{v}{U}\tan\Theta + \frac{p}{\rho U^2}\right\} = 0 \tag{14}$$

where the operator denotes differentiation in the streamwise direction. The quantity in brackets, which is proportional to the perturbation of stagnation pressure, therefore varies only from one streamline to the other, and the integral of equation 14 may be written

$$\frac{u}{U} + \frac{v}{U}\tan\Theta + (1+\tan^2\Theta)P = -\frac{1}{U}f(y - x\tan\Theta) \tag{15}$$

From equations 15 and 6 it is evident that f' is the vorticity and that it is transported along the undisturbed streamlines. In fact, from these same equations, the two velocity components may be written

$$\frac{u}{U} = -\mathcal{P} - \Theta \tan\theta - \frac{1}{(1+\tan^2\theta)U} f(y - x\tan\theta) \qquad 16$$

$$\frac{v}{U} = -\mathcal{P}\tan\theta + \Theta - \frac{\tan\theta}{(1+\tan^2\theta)U} f(y - x\tan\theta) \qquad 17$$

Now let us consider a single blade row, represented by an actuator surface located on the y-axis, for which the undisturbed uniform flow moves at an angle θ_1 upstream and θ_2 downstream in accordance with the diagram of Figure 4. There is a corresponding pressure rise Δp across the actuator. Let us suppose that, due either to a self induced or upstream induced disturbance, the cascade stalls over a fraction α of its circumference, and that the stall region is located as shown in Figure 4. According to the chosen stall model, the stall induces no disturbance to the relative discharge angle from the cascade, but the pressure rise Δp drops to (say) zero across the stalled portion. The condition on the pressure field is then that

$$\rho U^2(1+\tan^2\theta_2)\mathcal{P}_2(0,y) - \rho U^2(1+\tan^2\theta_1)\mathcal{P}_1(0,y)$$

$$= \begin{cases} -\Delta p\ ; & n - \frac{\alpha}{2} \le \frac{y}{2\pi R} \le n + \frac{\alpha}{2} \\ 0\ ; & \text{Elsewhere} \end{cases} \qquad 18$$

so that the difference between the solutions for the pressure fields upstream and downstream consists of positive jumps at the points $y = (n + \frac{\alpha}{2})\,2\pi R$ and equal negative jumps at the points $y = (n - \frac{\alpha}{2})\,2\pi R$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are harmonic functions, one may begin to construct a solution by obtaining the harmonic function that represents the value of $\mathcal{P}_2(0,y) - \mathcal{P}_1(0,y)$ and is regular elsewhere. Now the imaginary part of the function $\frac{1}{\pi}\log(z-i\eta)$ has a positive unit jump at $y=\eta$ for $x=0+$ and a corresponding negative jump for $x=0-$. The sort of function we require may then be constructed by situating the appropriate logarithmic singularities at $y=(n+\frac{\alpha}{2})2\pi R$ and the negative of such functions at $y=(n-\frac{\alpha}{2})2\pi R$ The analytic function

$$\mathcal{F}(z) = \frac{1}{\pi}\log\left\{\frac{\sinh\pi\left[\frac{z}{2\pi R}-i\frac{\alpha}{2}\right]}{\sinh\pi\left[\frac{z}{2\pi R}+i\frac{\alpha}{2}\right]}\right\}$$

posesses this property, since the hyperbolic sine in the numerator has simple zeros on the y-axis at $\frac{y}{2\pi R}=n-\frac{\alpha}{2}$, Figure 5, and the denominator has corresponding simple zeros on the y-axis at $\frac{y}{2\pi R}=n+\frac{\alpha}{2}$. The values of the real and imaginary parts along the y-axis, which are involved in satisfying matching conditions across the actuator line, are shown also in Figure 5. The real part, $F(x,y)$ is single valued but exhibits the expected logarithmic singularities at $z/2\pi R=(n\pm\frac{\alpha}{2})i$. The imaginary part, $G(x,y)$ with the branch cut taken along the y-axis, exhibits a jump of magnitude 2 in the intervals $n-\frac{\alpha}{2}\le\frac{y}{2\pi R}\le n+\frac{\alpha}{2}$ and is continuous elsewhere. Far upstream of the cascade, as $x\to-\infty$, $G(-\infty,y)$ has a constant value α and far downstream, as $x\to\infty$, $G(\infty,y)$ has a constant value $-\alpha$.

Now the solutions $\mathcal{P}_1(x,y)$ and $\mathcal{P}_2(x,y)$ for the pressure field can be constructed from a multiple of $G(x,y)$ and another harmonic function which must be regular except at the singularities along the imaginary axis. Hence the supplementary function can differ from a multiple of $F(x,y)$ by at most a constant. The flow angle and pressure far ahead of the cascade can vary by only constant magnitude from the undisturbed; these values correspond to variations in flow angle and pressure imposed far upstream. If we assume these to vanish, then the prsssure and angle perturbation fields become

$$(1+\tan^2\theta_1)P_1 = AF(x,y) + B[G(x,y)-\alpha] \qquad 19$$

$$(1+\tan^2\theta_1)\Theta_1 = -BF(x,y) + A[G(x,y)-\alpha] \qquad 20$$

$$(1+\tan^2\theta_2)P_2 = AF(x,y) + \left[\frac{\Delta p}{\rho U^2} - B\right]\left[G(x,y)+\alpha\right] - \frac{\Delta p}{\rho U^2}\alpha \qquad 21$$

$$(1+\tan^2\theta_2)\Theta_2 = -\left[\frac{\Delta p}{\rho U^2} - B\right]F(x,y) + A\left[G(x,y)+\alpha\right] + D\alpha \qquad 22$$

where the constants A, B, D remain to be determined. The available matching conditions are (i) that the axial velocity is continuous across the cascade and (ii) that the relative discharge angle is given in terms of the inlet angle according to Figure 3. For the present consideration we shall assume that the relative discharge angle is fixed, independent of the inlet angle. The representations of the axial velocities upstream and downstream of the cascade may be written

$$\frac{u_1}{U} = -P_1 - \Theta_1 \tan\theta_1 \qquad 23$$

$$\frac{u_2}{U} = -\left(P_2 + \frac{\Delta p}{\rho U^2}\frac{\alpha}{1+\tan^2\theta_2}\right) - \left(\Theta_2 - D\frac{\alpha}{1+\tan^2\theta_2}\right)\tan\theta_2$$

$$+ a_2 F(0, y - x\tan\theta_2) + b_2\left[G(0+, y - x\tan\theta_2) + \alpha\right] \qquad 24$$

Where u_2 has been supplied with an additive constant which assures that u_2 averages to zero far downstream. The condition that the relative discharge angle from the cascade be invarient requires expressing the relative flow angle perturbation in terms of perturbations seen from the steady flow field. Referring to equations 1 and 2, which relate these flow angles,

$$\tan(\theta+\vartheta') - \tan(\beta+\beta') = \begin{cases} \dfrac{\omega R}{U+u'} \ ; & \text{STATOR} \\[2ex] \dfrac{\omega R - \Omega R}{U+u'} \ ; & \text{ROTOR} \end{cases} \qquad 25$$

so that to the first order in small disturbances,

$$(1+\tan^2\beta)\,\beta' - (1+\overline{\tan^2\theta})\,\vartheta' = \begin{cases} \left(\frac{\omega R}{U}\right)\frac{u'}{U} & ; \ \text{STATOR} \\ \left(\frac{\omega R}{U} - \frac{\Omega R}{U}\right)\frac{u'}{U} & ; \ \text{ROTOR} \end{cases} \qquad 26$$

The condition that the outlet angle be unchanged is therefore

$$\left(1+\tan^2\theta_2\right)\Theta_2(0,y) + \begin{Bmatrix} \frac{\omega R}{U} \\ \frac{\omega R}{U} - \frac{\Omega R}{U} \end{Bmatrix} \frac{u_2}{U}(0,y) = 0 \qquad 27$$

which we write in the form

$$\Theta_2(0,y) + K\,\frac{u_2}{U}(0,y) = 0 \qquad 28$$

where

$$K = \begin{cases} \dfrac{\omega R/U}{1+\tan^2\theta_2} & ; \ \text{STATOR} \\ \dfrac{\omega R/U - \Omega R/U}{1+\tan^2\theta_2} & ; \ \text{ROTOR} \end{cases} \qquad 29$$

Now all of the field quantities are represented by equations 19 through 24 and substitution into the two matching conditions provides relations among the unknown coefficients. In fact, the coefficients of $F(0,y)$ and $[G(0,y)+\alpha]$ must vanish identically because these two functions are orthogonal over the range $0 \le y \le 2\pi R$ Substituting into the expression $\frac{u_1}{U}(0,y) = \frac{u_2}{U}(0,y)$ for continuity of axial velocity leads to the algebraic equations

$$\left(\frac{-1}{1+\tan^2\theta_1} + \frac{1}{1+\tan^2\theta_2}\right)A + \left(\frac{\tan\theta_1}{1+\tan^2\theta_1} + \frac{\tan\theta_2}{1+\tan^2\theta_2}\right)B - a_2 = \frac{\tan\theta_2}{1+\tan^2\theta_2}\,\frac{\Delta P}{\rho U^2} \qquad 30$$

$$\left(\frac{-\tan\theta_1}{1+\tan^2\theta_1}-\frac{\tan\theta_2}{1+\tan^2\theta_2}\right)A+\left(\frac{-1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right)B+b_2=\frac{1}{1+\tan^2\theta_2}\,\frac{\Delta p}{\rho U^2} \qquad 31$$

Similarly, substitution into the condition on the cascade discharge, angle, equation 28, gives the second pair of algebraic equations.

$$\frac{-K}{1+\tan^2\theta_2}A+\frac{1-K\tan\theta_2}{1+\tan^2\theta_2}B+Ka_2=\frac{1-K\tan\theta_2}{1+\tan^2\theta_2}\,\frac{\Delta p}{\rho U^2} \qquad 32$$

$$\frac{1-K\tan\theta_2}{1+\tan^2\theta_2}A+\frac{K}{1+\tan^2\theta_2}B+Kb_2=\frac{K}{1+\tan^2\theta_2}\,\frac{\Delta p}{\rho U^2} \qquad 33$$

These equations constitute a linear set for A, B, a_2, b_2 and may be written, after some manipulation, in the matrix form

$$\begin{bmatrix}
\left(\frac{-1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right) & \left(\frac{\tan\theta_1}{1+\tan^2\theta_1}+\frac{\tan\theta_2}{1+\tan^2\theta_2}\right) & -1 & 0\\
\left(\frac{-\tan\theta_1}{1+\tan^2\theta_1}-\frac{\tan\theta_2}{1+\tan^2\theta_2}\right) & \left(\frac{-1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right) & 0 & 1\\
\frac{-K}{1+\tan^2\theta_2} & \left(\frac{K\tan\theta_1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right) & 0 & 0\\
\left(\frac{K\tan\theta_1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right) & \frac{K}{1+\tan^2\theta_2} & 0 & 0
\end{bmatrix}
\begin{bmatrix} A\\ B\\ a_2\\ b_2\end{bmatrix}
=
\begin{bmatrix}\frac{\tan\theta_2}{1+\tan^2\theta_2}\\ \frac{1}{1+\tan^2\theta_2}\\ \frac{1}{1+\tan^2\theta_2}\\ 0\end{bmatrix} \qquad 34$$

The determinant of the matrix on the left,

$$\Delta=\left(\frac{K}{1+\tan^2\theta_1}\right)^2+\left(\frac{K\tan\theta_1}{1+\tan^2\theta_1}+\frac{1}{1+\tan^2\theta_2}\right)^2 \qquad 35$$

is non-vanishing and the resulting values of A, B, a_2, b_2 are proportional to $\frac{\Delta p}{\rho U^2}$, the pressure loss coefficient at stall. The results will not be written down explicitly now because they will emerge later in a more general form. It is well to note, however, that because $a_2 \neq 0$, $b_2 \neq 0$, the flow far downstream of the cascade is permanently distorted and corresponds to a wake generated by the vorticity sheet from the stalled zone.

The physical problem of the single blade row is still incomplete. It has been assumed that the cascade is stalled over a certain portion of its periphery but the conditions which lead to the stall have not been considered. We must examine two possibilities: i) the stall is induced by a flow distortion originating upstream of the cascade and ii) the stall is self-induced. The case of the self-induced or "propagating stall" will be presented in this section; the stall induced by flow distortions will be treated subsequently.

Now whether the cascade stalls locally is determined, according to Figure 2, by the local inlet angle β_1. In the absence of disturbance the uniform inlet angle β_1 is assumed less than the stall angle β_1^*, and in the presence of the stall, the perturbation $\beta_1'(0,y)$ must relate appropriately to the angle increment $\beta_1^* - \beta_1$ required to stall the airfoil. Using the relationship we have found between absolute and relative perturbation angles, we have

$$(1+\tan^2\beta_1)\beta_1'(0,y) = (1+\tan^2\theta_1)\Theta_1(0,y) + (1+\tan^2\theta_2)K\frac{u_1}{U}(0,y)$$

Now using the solution forms we have found for $\Theta_1(x,y)$ and $\frac{u_1}{U}(x,y)$

$$(1+\tan^2\beta_1)\beta_1'(0,y) = \left[(1+\tan^2\theta_1) - (1+\tan^2\theta_2)K\tan\theta_1\right]\Theta_1 - (1+\tan^2\theta_2)K\mathrm{P}_1$$

$$= \frac{-1}{1+\tan^2\theta_1}\left[\left(1+\tan^2\theta_1 - (1+\tan^2\theta_2)K\tan\theta_1\right)B + (1+\tan^2\theta_2)KA\right]F(0,y)$$

$$+ \frac{1}{1+\tan^2\theta_1}\left[\left((1+\tan^2\theta_1) - (1+\tan^2\theta_2)K\tan\theta_1\right)A - (1+\tan^2\theta_2)KB\right]\left[G(0-,y)-\alpha\right] \qquad 36$$

The stable, self-induced stall cell requires that the inlet flow angle change from a value below β_1^* to a value equal to or greater than β_1^* at the edges of the cell and this condition can be satisfied only if the coefficient of $F(0,y)$ vanishes. Thus

$$(1+\tan^2\theta_2)KA + \left[1+\tan^2\theta_1 - (1+\tan^2\theta_2)K\tan\theta_1\right]B = 0 \qquad 37$$

This additional condition on the constants A and B may be taken together with the fourth of equation 34, another homogeneous condition, to require that, for a non-trivial solution,

$$\begin{vmatrix} \left(\dfrac{K\tan\theta_1}{1+\tan^2\theta_1} + \dfrac{1}{1+\tan^2\theta_2}\right) & \dfrac{K}{1+\tan^2\theta_1} \\ \dfrac{K}{1+\tan^2\theta_1} & \left(\dfrac{-K\tan\theta_1}{1+\tan^2\theta_1} + \dfrac{1}{1+\tan^2\theta_2}\right) \end{vmatrix} = 0 \qquad 38$$

which constitutes a characteristic value condition for K. Physically, this states that the self-induced stall cell must move with a certain speed in order that the flow field satisfy the condition on the inlet angle approaching the stall cell. This free or self-induced stall cell has become known as propagating stall because it must move along the blade row at this speed to assure the stability of the leading trailing edges of the stall cell. The value of the determinant in equation 38 gives the characteristic value as

$$(1+\tan^2\theta_2)K \equiv \begin{Bmatrix} \dfrac{\omega R}{U} \\ \dfrac{\omega R}{U} - \dfrac{\Omega R}{U} \end{Bmatrix}$$

$$= \sqrt{1+\tan^2\theta_1} = -\frac{1}{\sin 2\beta_1} \qquad 39$$

where the relation between relative and absolute flow angles has been used to obtain the final expression. Here β_1 is essentially the relative inlet stalling angle of the blade row in question and ωR is the velocity of stall propagation.

Note that the value of K determined for the self-induced stall does not make the matrix in equation 34 singular but allows determination of the complete set of coefficients, in particular, the values of a_2 and b_2 which give the stall wake that appears far downstream from the blade row.

4. SINGLE BLADE ROW; GENERAL ANALYSIS

The analysis of section 3 has shown that the vorticity shed from a stalled blade row produces a wake that deforms the flow field far downstream of that blade row. It follows, moreover, that subsequent blade rows, in relative motion to the stalled region, will experience strong deviations of the relative inlet angle which may induce stall in that blade row also. In general, then, solution of the multistage problem will require knowledge of the behavior of single stages with a deformed input. It is the aim of this section to find such a solution, suitable for use in multistage analysis.

In the previous section it appeared that the axial velocity profile from a stalled single blade row, consisted of a linear combination of $F(0, y - x\tan\theta)$ and $G(0+, y - x\tan\theta) + \alpha$ even though the flow angle perturbation vanished far downstream. Therefore a distortion of this sort will appear upstream of the next blade row. In general, then, a blade row will have some linear combination of $F(0, y - x\tan\theta)$ and $G(0-, y - x\tan\theta) - \alpha$ upstream; a solution that accommodates this initial condition will permit also a distortion of this type to be prescribed ahead of the entire compressor.

To satisfy these requirements, the pressure and angle of perturbations may be written

$$(1+\tan^2\theta_1)\,P_1 = A\,F(x,y) + B[G(x,y) - \alpha] + C\alpha \qquad 40$$

$$(1+\tan^2\theta_1)\,\Theta_1 = -B\,F(x,y) + A[G(x,y) - \alpha] + D_1\alpha \qquad 41$$

$$(1+\tan^2\theta_2)\,P_2 = A\,F(x,y) + \left[\frac{\Delta p}{\rho U^2} - B\right][G(x,y) + \alpha] - \frac{\Delta p}{\rho U^2}\alpha + C\alpha \qquad 42$$

$$(1+\tan^2\theta_2)\,\Theta_2 = -\left[\frac{\Delta p}{\rho U^2} - B\right]F(x,y) + A[G(x,y) + \alpha] + D_2\alpha \qquad 43$$

while the corresponding axial velocity distributions are

$$\frac{u_1}{U} = -\left(\beta_1 - \frac{C}{1+\tan^2\theta_1}\alpha\right) - \left(\Theta_1 - \frac{D_1}{1+\tan^2\theta_1}\alpha\right)\tan\theta_1$$
$$+ a_1 F(0, y - x\tan\theta_1) + b_1\left[G(0-, y - x\tan\theta_1) - \alpha\right] \tag{44}$$

$$\frac{u_2}{U} = -\left(\beta_2 + \frac{\Delta p}{\rho U^2}\frac{\alpha}{1+\tan^2\theta_2} - C\frac{\alpha}{1+\tan^2\theta_2}\right) - \left(\Theta_2 - \frac{D_2}{1+\tan^2\theta_2}\alpha\right) \tag{45}$$
$$+ a_2 F(0, y - x\tan\theta_2) + b_2\left[G(0+, y - x\tan\theta_2) + \alpha\right]$$

The prescribed conditions ahead of the blade row determine the values of a_1, b_1, C, D_1 and α and the values of A, B, a_2 b_2 and D_2 are required. These constants are evaluated as a consequence of matching the axial velocity perturbations.

$$u_1(0,y) = u_2(0,y) \tag{46}$$

and fixing the relative discharge angle

$$\Theta_2(0,y) + K\frac{u_2}{U}(0,y) = 0 \tag{47}$$

The pressure loss caused by stall has already been accounted for in constructing the solutions. Now a prescribed inlet distortion, fixed by the values of a_1 and b_1, may or may not induce stall over a portion $2\pi R\alpha$ of the blade row periphery. This question is determined by the relative inlet angle; if stall is indicated, Δp in equations 40-45 is set equal to the value appropriate for the blade row, if not, Δp is set equal to zero.

Another feature that appears when the stall is forced by the inlet distortion is that the value of K is prescribed. For a stationary distortion, such as strut or a separated inlet, $\omega = 0$, but in the unusual circumstances of a moving disturbance, which may occur with an ingested vortex, the value of ω is assumed known. This will mean that, in general, the inlet angle distortion, will not be a simple square wave $G(o-,y)$ but will contain the singularities characterizing $F(o,y)$. The condition for stall in this circumstance will be discussed later.

Substitution into conditions 46 and 47 leads, in a straightforward manner, to the result that $D_2 = 0$ and a set of linear algebraic equations for the unknown quantities A, B, a_2, b_2 . These are identical with the equations 34 except for additional terms on the right-hand side

$$\begin{bmatrix} \frac{-1}{1+\tan^2\theta_1} + \frac{1}{1+\tan^2\theta_2} & \frac{\tan\theta_1}{1+\tan^2\theta_1} + \frac{\tan\theta_2}{1+\tan^2\theta_2} & -1 & 0 \\ \frac{-\tan\theta_1}{1+\tan^2\theta_1} - \frac{\tan\theta_2}{1+\tan^2\theta_2} & \frac{-1}{1+\tan^2\theta_1} - \frac{1}{1+\tan^2\theta_2} & 0 & 1 \\ \frac{-K}{1+\tan^2\theta_1} & \frac{K\tan\theta_1}{1+\tan^2\theta_1} + \frac{1}{1+\tan^2\theta_2} & 0 & 0 \\ \frac{K\tan\theta_1}{1+\tan^2\theta_1} + \frac{1}{1+\tan^2\theta_2} & \frac{K}{1+\tan^2\theta_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ a_2 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\tan\theta_2}{1+\tan^2\theta_2} \\ \frac{1}{1+\tan^2\theta_2} \\ \frac{1}{1+\tan^2\theta_2} \\ 0 \end{bmatrix} \frac{\Delta p}{\rho U^2} + \begin{bmatrix} -1 \\ 0 \\ -K \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ K \end{bmatrix} b_1 \qquad 48$$

The inversion is easily effected and can be expressed in the form

$$\begin{bmatrix} A \\ B \\ a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \mu_{11} & \mu_{12} & \mu_{23} \\ \mu_{21} & \mu_{22} & \mu_{23} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ \frac{\Delta p}{\rho U^2} \end{bmatrix} \qquad 49$$

where ν_{ij} and μ_{ij} are functions of θ_1, θ_2, K and are tabulated in the Appendix.

Now a_1 and b_1 are given through prescription of the flow far upstream but $\frac{\Delta p}{\rho U^2}$ is either a known value or zero according to the approach angle relative to the blade row. The relative inlet angle perturbation, given by equation 36, becomes for the present example,

$$
\begin{aligned}
&(1+\tan^2\beta_1)\,\beta_1'(0,y) \\
&= -\left\{\frac{1}{1+\tan^2\theta_1}(1+\tan^2\theta_2)KA + \left[1-\frac{\tan\theta_1}{1+\tan^2\theta_1}(1+\tan^2\theta_2)K\right]B + (1+\tan^2\theta_2)Ka_1\right\}F(0,y) \\
&- \left\{-\left[1-\frac{\tan\theta_1}{1+\tan^2\theta_1}(1+\tan^2\theta_2)K\right]A - \frac{1}{1+\tan^2\theta_1}(1+\tan^2\theta_2)KB - (1+\tan^2\theta_2)K\,b_1\right\}\left[G(0-,y)-\alpha\right]
\end{aligned}
\qquad 50
$$

Because the $\lim_{\varepsilon\to 0}\int_{\varepsilon-n\alpha}^{n\alpha-\varepsilon} F(0,y)\,dy = 0$ this does not, on the average, contribute to the inlet angle in the region under consideration. The function $G(0-,y)-\alpha$ however, equals $1-\alpha$ in the stalled region and $-\alpha$ in the region exterior to the stall. In detail, then, the inlet angle perturbation is, in the distorted region

$$
\begin{aligned}
\tilde{\beta}_1'(0,y) = \frac{(1-\alpha)K}{1+\tan^2\beta_1}\Bigg\{&\frac{-2}{(1+\tan^2\theta_1)(1+\tan^2\theta_2)\,\Delta}\left(\frac{\Delta p}{\rho U^2}\right) \\
&+ \frac{2K}{(1+\tan^2\theta_1)\,\Delta}\,a_1 + \left[\frac{1-K^2(1+\tan^2\theta_1)}{(1+\tan^2\theta_2)\,\Delta} + (1+\tan^2\theta_1)\right]b_1\Bigg\}
\end{aligned}
\qquad 51
$$

where Δ is the determinant given in the Appendix. Then if

$$
\tilde{\beta}_1'(0,y) \geq \beta_1^* - \beta_1 \qquad 52
$$

the cascade will stall in the region $-n R\alpha \leq y \leq n R\alpha$ in which the upstream distortion impinges upon it and the solution is correct as it stands. If, however, the inequality 52 is not satisfied then the solution given by equations 49-52 must be evaluated with $\frac{\Delta p}{\rho U^2}$ set to zero. This condition also determines the pressure loss far downstream from the cascade, according to equation 42, as $\frac{\Delta p}{\rho U^2}\alpha$ or zero.

5. ANALYSIS OF MULTISTAGE COMPRESSOR WITH WIDE SPREAD BLADE ROWS

The results of the last two sections demonstrate that a compressor blade row, represented by an actuator surface, which is subjected to a distortion in axial velocity profile that is a linear combination of $F(0,y)$ and $G(0-,y)-\alpha$, produces a distorted discharge flow which is another linear combination of the same two functions. This general result holds, whether or not the cascade in question stalls as a result of the distorted input. It follows, then, that we can construct the flow in a multistage compressor, represented in this two-dimensional approximation, from linear combinations of the two fundamental profiles given above. The situation holds regardless of how many of the various blade rows stall.

Number the blade rows from the inlet, the first of which may be the inlet vanes, the successive ones may alternate rotor and stator rows. Number the region ahead of the n^{th} blade row by the subscript n, that is $u_n, \beta_n', P_n, \Theta_n$ and the region downstream by the subscript $n+1$, that is $u_{n+1}, \beta_{n+1}', P_{n+1}, \Theta_{n+1}$. Associated with each of these blade rows is a stalling inlet angle β_n^*, a fixed relative outlet angle β_{n+1}, a pressure loss at stall, Δp_n, and a value K_n of the blade motion parameter. For flow fields generated by a stationary upstream disturbance, the values of K_n are

$$K_n = \left\{ \begin{array}{ll} 0 & ; \quad \text{STATOR} \\ \dfrac{-\Omega R/U}{1+\tan^2\Theta_2} & ; \quad \text{ROTOR} \end{array} \right. \qquad (53)$$

We presume the unperturbed state to be known and this means, in detail, that values of $U, \Omega R, \Theta_n, \beta_n$ are known. The perturbations we wish to compute are caused by a distortion upstream of the compressor inlet described by the values of a_1 and b_1 ; generally this takes the form of a distortion to the axial velocity profile given by

$$\frac{u_1}{U} = a_1 F(0, y) + b_1 [G(0-, y) - \alpha] \tag{54}$$

We shall assume, for the purpose of this demonstration, that the pressure and angle disturbances vanish far upstream and that $\theta_1 = 0$. Note specifically that there is no perturbation to the total flow rate, that is

$$\int_0^{2\pi R} u_1 \, dy = 0$$

so that the mean or reference axial velocity is undisturbed.

Now the single blade row analysis of the previous section demonstrated that, for a blade row with prescribed characteristics, the flow distortion downstream was determined by the flow distortion entering from upstream along with the knowledge of whether or not the modified inlet flow stalled a section of the blade row in question. The relationship of the distortion in the discharge to the conditions ahead of the blade row is given by expressions of the type shown in the third and fourth of equations 49 and the condition for stall is determined by considerations similar to those of equations 51 and 52. There is no difficulty in generalizing this scheme to one that will apply to any blade row of a multistage compressor.

The generalized calculation is carried out in the following steps. First suppose the n^{th} blade row is stalled over the sections $-\alpha\pi R \le y \le \alpha\pi R$. The coefficients of the downstream perturbation and the perturbation to inlet angle in the stalled region may be written in the matrix form

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ \tilde{\beta}_n' \end{bmatrix} = \begin{bmatrix} \mu_{11}(\theta_n, \theta_{n+1}, K_n) & - & - \\ - & & - - \\ - & & - \; \mu_{33}(\theta_n, \theta_{n+1}, K_n) \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ \frac{\Delta p_n}{\rho U^2} \end{bmatrix}$$

The nine coefficients of this matrix are given in the Appendix. Second, determine whether the value of $\tilde{\beta}_n'$ is greater or less than the known value of $\beta_n^* - \beta_n$. If $\tilde{\beta}_n' \geq (\beta_n^* - \beta_n)$ then the calculation above is correct and the resulting information permits proceeding to the next blade row. If, however, $\tilde{\beta}_n' < (\beta_n^* - \beta_n)$ the above calculation must be repeated with the element $\Delta p_n / \rho U^2$ replaced by zero. Then this new information permits proceeding to the next blade row. The loss in pressure rise across the stage, it will be recalled, is $\alpha \frac{\Delta p_n}{\rho U^2}$ or zero depending upon whether the blade row is stalled, and these increments will accumulate as we pass through the compressor.

One further point should be mentioned in order to clarify the application of this procedure; the circumferential coordinate of each blade row has been adjusted so that the distorted region appears symmetrically located with respect to the local y -axis. To recover the physical flow field, the resulting profiles and angles should be applied with respect to an undisturbed streamline passing through the machine.

6. CONCLUDING REMARKS

The example that we have examined in some detail has demonstrated that, within the small perturbation theory, the exact solution consists of a linear combination of two basic flow fields. In particular, the axial and tangential velocity profiles constitute a two-parameter family where the parameters are functions of axial position through the compressor.

It is clear then, that if we chose to represent the solution in terms of this two-parameter family for an integral method, and employ the actuator surface characteristics of the preceding analysis, the integral method will yield the same solution. For example, we represent the axial velocity in the form

$$\frac{u_1}{U} = L_1 F(x,y) + M_1 [G(x,y) - \alpha] + l_1 F(0, x - y \tan\theta_1) + m_1 [G(0-, y - x \tan\theta_1) - \alpha]$$

where the parameters L_1, M_1 are unknown but l_1, m_1 are given. The quantities v_1/U and $p_1/\rho U^2$ and the corresponding quantities downstream of the blade row have corresponding representations. Integration of the equations of continuity and motion in the peripheral direction establishes the relationship between the parameters that occur in u, v and p ; detailed matching across the actuator surface establishes the relationship between the parameters upstream and downstream. Note that the matching across the actuator surface corresponds to applying a geometric constraint representing a blade shape or to a force field representing a distributed blade loading. As a result, the differential equations that describe the variation of parameters along the compressor when the load is continuously distributed, degenerate to difference equations for the compressor represented by a succession of actuator surfaces. These difference equations correspond to the matrix relations given in equations 48, or more precisely, equations 55.

REFERENCES

1. Fuhs, Allen E., "Distortion Induced Engine Instability," AGARD Lecture Series No. 72, October 1974, Technical Editing and Reproduction, London.

2. ——, "Unsteady Phenomena in Turbomachinery," Proceedings of 46th Meeting, Propulsion and Energetics Panel, Naval Postgraduate School, Monterey, California, Sept. 1975.

3. Rannie, W. D. and Marble, Frank E., "Unsteady Flows in Axial Turbomachines," Comptes Rendus des Journees Internationales de Sciences Aeronautiques, 27 May 1957, ONERA, Paris.

4. Katz, Robert, "Performance of Axial Compressors with Asymmetric Inlet Flows," Final Report, AFOSR Contract 18(600)-178, California Institute of Technology, Daniel and Florence Guggenheim Jet Propulsion Center, June 1958.

5. Marble, Frank E. and Heiser, William H., "Some Preliminary Experiments Concerning the Effects of Strongly Distorted Inlet Flow on the Performance of an Axial Compressor," Final Report, Contract No. AF49(638)-497, California Institute of Technology, Pasadena, California, June 1960.

6. Marble, Frank E., "Propagation of Stall in a Compressor Blade Row," Journal of the Aeronautical Sciences, Vol. 22, No. 8, August 1955.

APPENDIX

The determinant of the matrix on the left-hand side of equation 48 is

$$\Delta_n = \left(\frac{K_n}{1+\tan^2\theta_n}\right)^2 + \left(\frac{K_n \tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}}\right)^2 \qquad \text{A-1}$$

The coefficients in the equations 49 are

$$\nu_{11} = \frac{1}{\Delta_n} \frac{K_n^2}{1+\tan^2\theta_n} \qquad \text{A-2}$$

$$\nu_{12} = \frac{1}{\Delta_n} K_n \left(\frac{K_n \tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}}\right) \qquad \text{A-3}$$

$$\nu_{13} = \frac{1}{\Delta_n} \frac{K_n}{1+\tan^2\theta_n} \qquad \text{A-4}$$

$$\nu_{21} = -\frac{1}{\Delta_n} K_n \left(\frac{K_n \tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}}\right) \qquad \text{A-5}$$

$$\nu_{22} = \frac{1}{\Delta_n} \frac{K_n^2}{1+\tan^2\theta_n} \qquad \text{A-6}$$

$$\nu_{23} = \frac{1}{\Delta_n} \left(\frac{1}{1+\tan^2\theta_{n+1}}\right)\left(\frac{K_n \tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}}\right) \qquad \text{A-7}$$

$$\mu_{11} = -\frac{1}{\Delta_n}\begin{vmatrix} -\frac{1}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & \frac{\tan\theta_n}{1+\tan^2\theta_{n+1}}+\frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} & 1 \\ \frac{-K_n}{1+\tan^2\theta_n} & \frac{K_n\tan\theta_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & K_n \\ \frac{K_n\tan\theta_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & \frac{K_n}{1+\tan^2\theta_n} & 0 \end{vmatrix} \quad \text{A-8}$$

$$\mu_{21} = -\frac{K_n}{\Delta_n}\begin{vmatrix} \frac{-\tan\theta_n}{1+\tan^2\theta_n}-\frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} & -\frac{1}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} \\ \frac{K_n\tan\theta_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & \frac{K_n}{1+\tan^2\theta_n} \end{vmatrix} \quad \text{A-9}$$

$$\mu_{31} = \frac{1}{\Delta_n}\;\frac{2(1-\alpha)K_n^2}{(1+\tan^2\beta_n)(1+\tan^2\theta_n)} \quad \text{A-10}$$

$$\mu_{12} = \frac{K_n}{\Delta_n}\begin{vmatrix} -\frac{1}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & \frac{\tan\theta_n}{1+\tan^2\theta_n}+\frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} \\ \frac{-K_n}{1+\tan^2\theta_n} & \frac{K_n\tan\theta_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} \end{vmatrix} \quad \text{A-11}$$

$$\mu_{22} = -\frac{1}{\Delta_n}\begin{vmatrix} \frac{-\tan\theta_n}{1+\tan^2\theta_n}-\frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} & \frac{-1}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & -1 \\ \frac{-K_n}{1+\tan^2\theta_n} & \frac{K_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & 0 \\ \frac{K_n\tan\theta_n}{1+\tan^2\theta_n}+\frac{1}{1+\tan^2\theta_{n+1}} & \frac{K_n}{1+\tan^2\theta_n} & K_n \end{vmatrix} \quad \text{A-12}$$

$$\mu_{32} = \frac{1}{\Delta_n}\left[\frac{(1-\alpha)K_n}{1+\tan^2\beta_n}\right]\left[\frac{1-K_n^2(1+\tan^2\theta_n)}{1+\tan^2\theta_{n+1}}\right] + (1-\alpha)K_n\left[\frac{1+\tan^2\theta_n}{1+\tan^2\beta_n}\right] \quad \text{A-13}$$

$$\mu_{13} = \frac{1}{\Delta_n} \begin{vmatrix} \frac{-1}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & \frac{\tan\theta_n}{1+\tan^2\theta_n} + \frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} & \frac{\tan\theta_n}{1+\tan^2\theta_n} \\ \frac{-K_n}{1+\tan^2\theta_n} & \frac{K_n\tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & \frac{1}{1+\tan^2\theta_n} \\ \frac{K_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & \frac{K_n}{1+\tan^2\theta_n} & 0 \end{vmatrix} \qquad \text{A-14}$$

$$\mu_{23} = \frac{1}{(1+\tan^2\theta_n)\Delta_n} \begin{vmatrix} \frac{-\tan\theta_n}{1+\tan^2\theta_n} - \frac{\tan\theta_{n+1}}{1+\tan^2\theta_{n+1}} & \frac{-1}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & 1 \\ \frac{-K_n}{1+\tan^2\theta_n} & \frac{K_n\tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & 1 \\ \frac{K_n\tan\theta_n}{1+\tan^2\theta_n} + \frac{1}{1+\tan^2\theta_{n+1}} & \frac{K_n}{1+\tan^2\theta_n} & 0 \end{vmatrix} \qquad \text{A-15}$$

$$\mu_{33} = -\frac{1}{\Delta_n}\left[\frac{2(1-\alpha)K_n}{1+\tan^2\beta_n}\right]\left[\frac{1}{(1+\tan^2\theta_n)(1+\tan^2\theta_{n+1})}\right] \qquad \text{A-16}$$

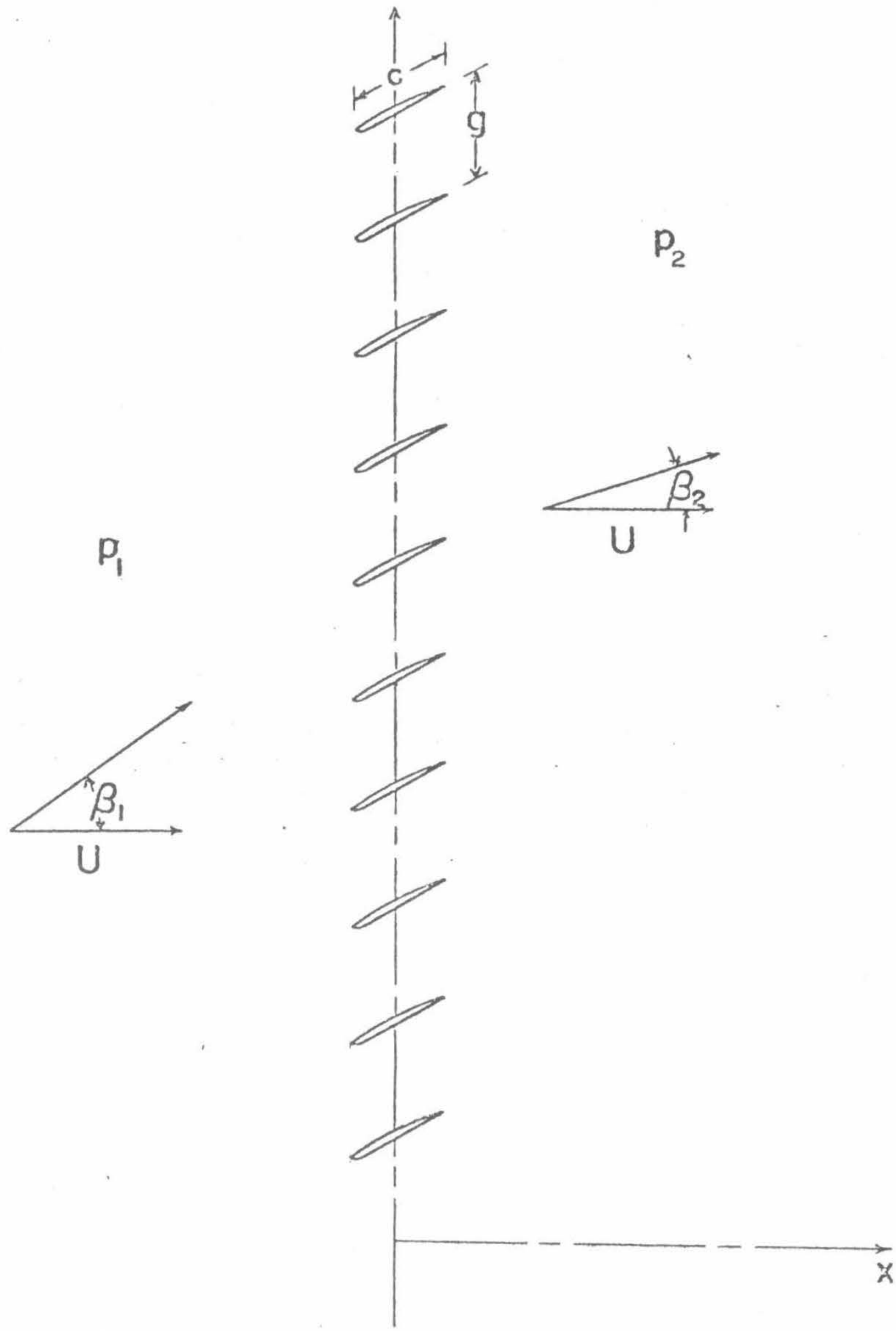

Figure 1. Diagram of actuating line.

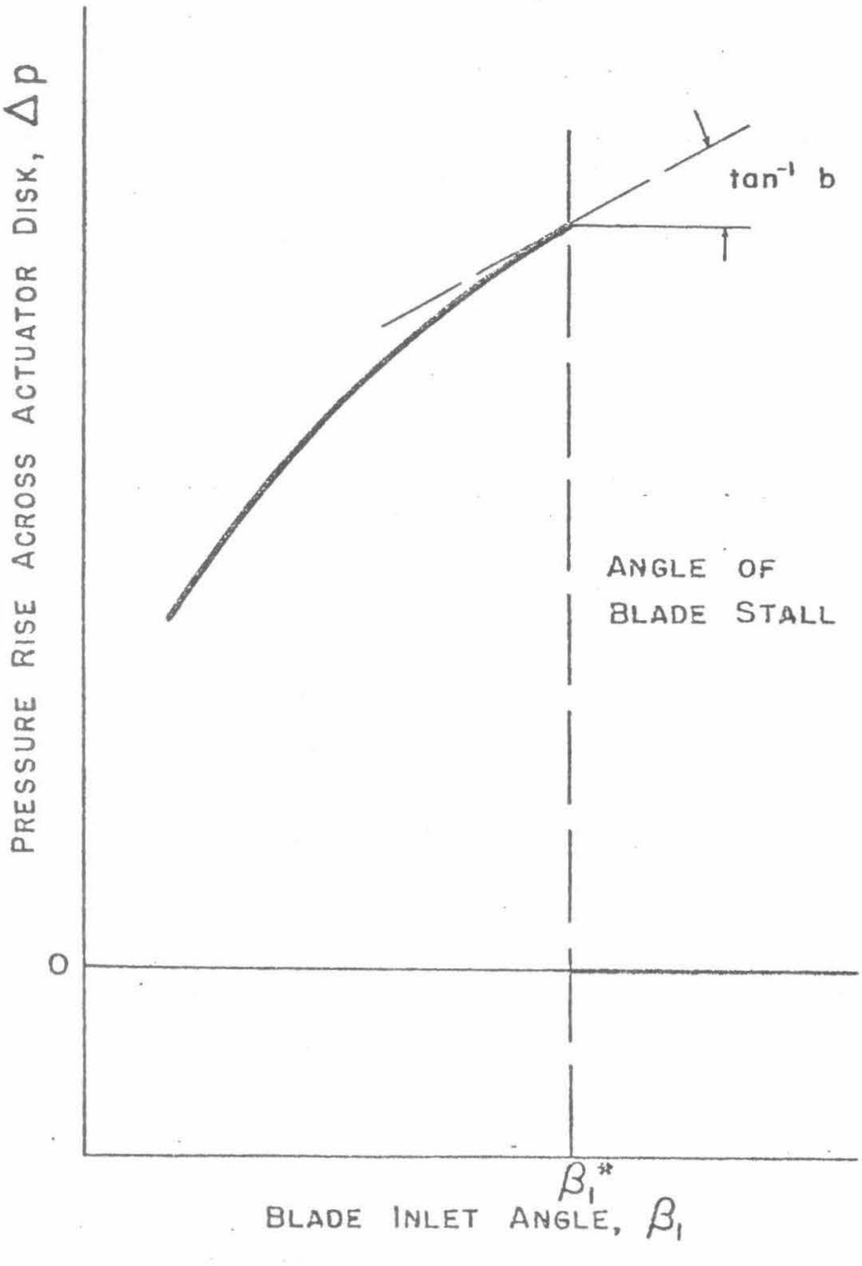

Figure 2. Pressure rise as a function of inlet angle.

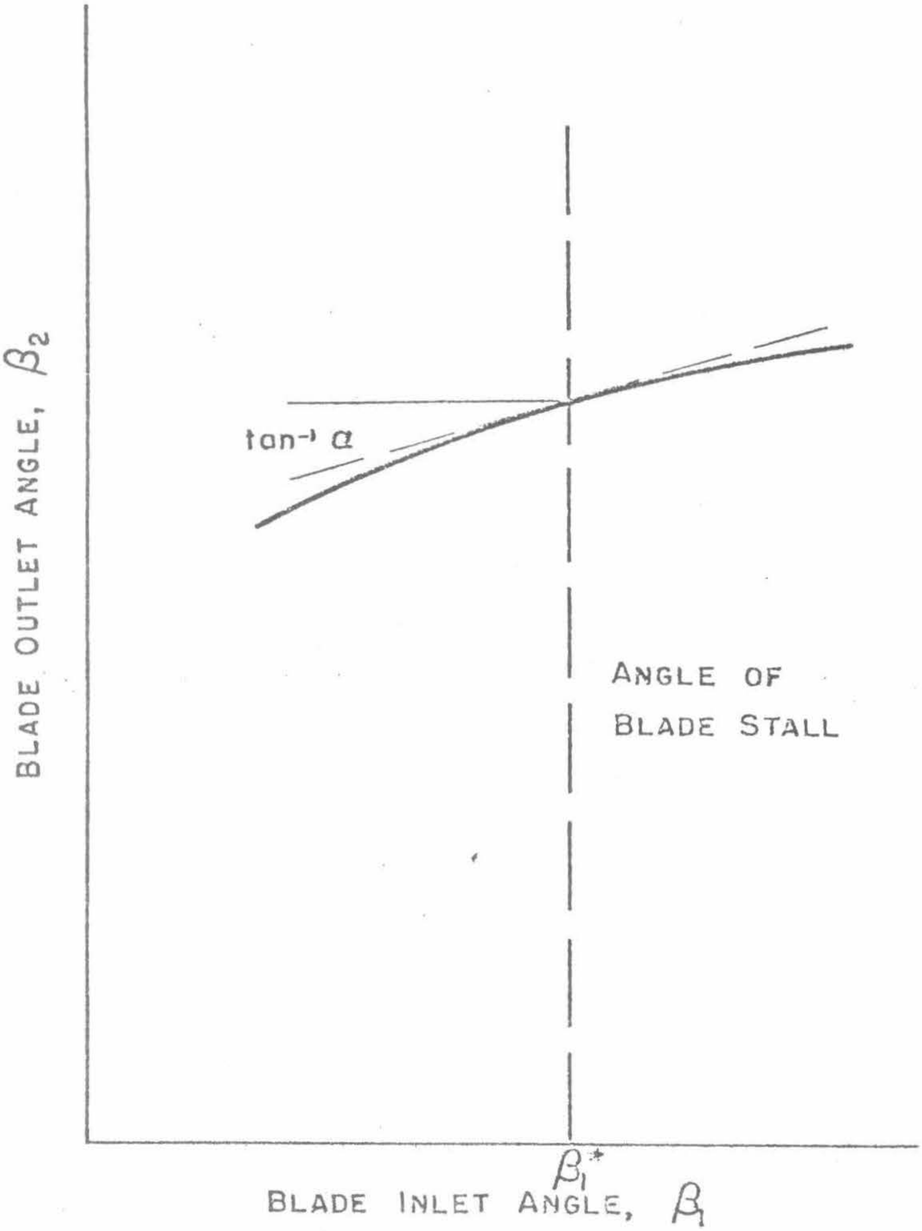

Figure 3. Discharge angle as a function of inlet angle.

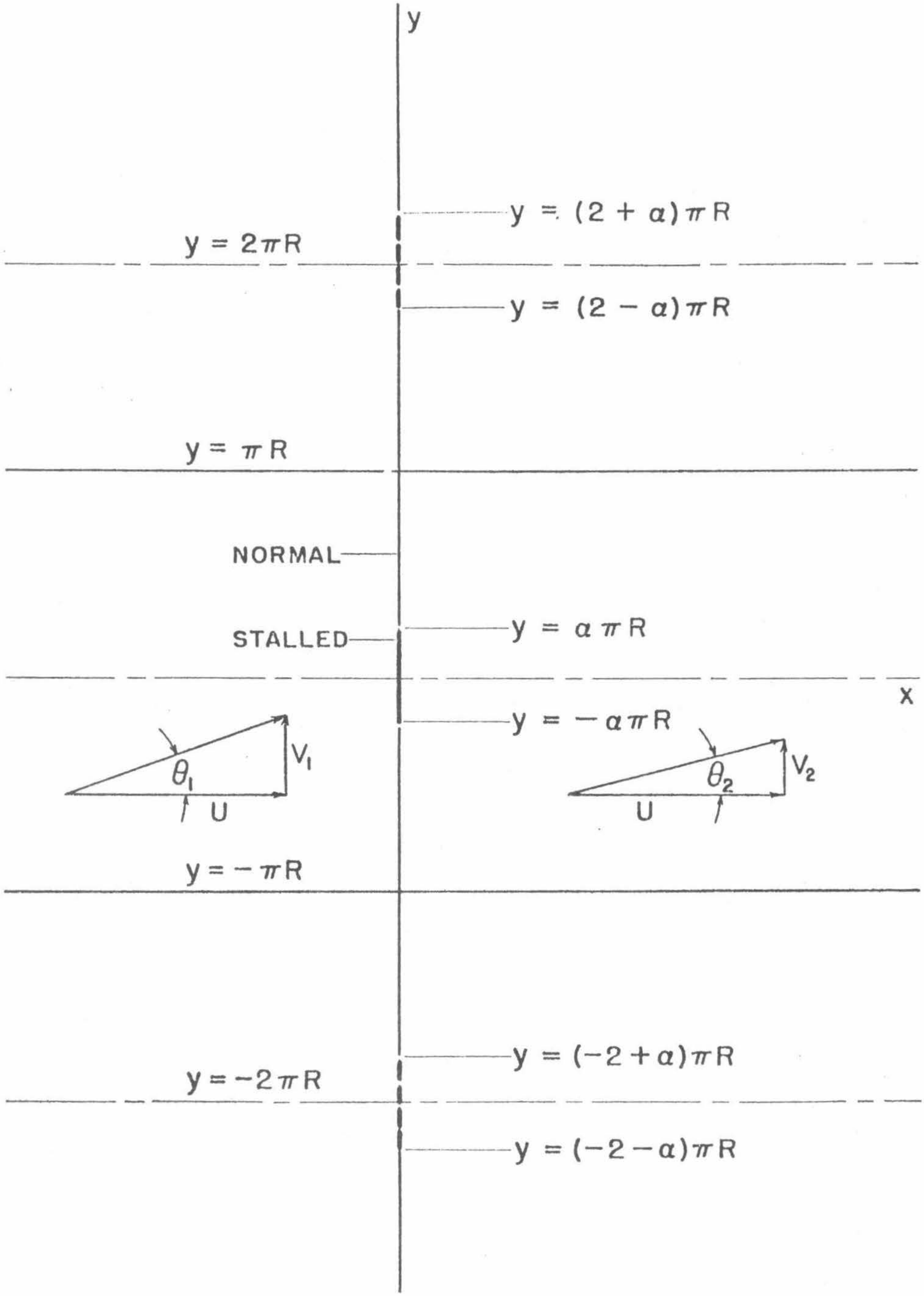

Figure 4. Pattern of stalled and unstalled regions on actuating line.

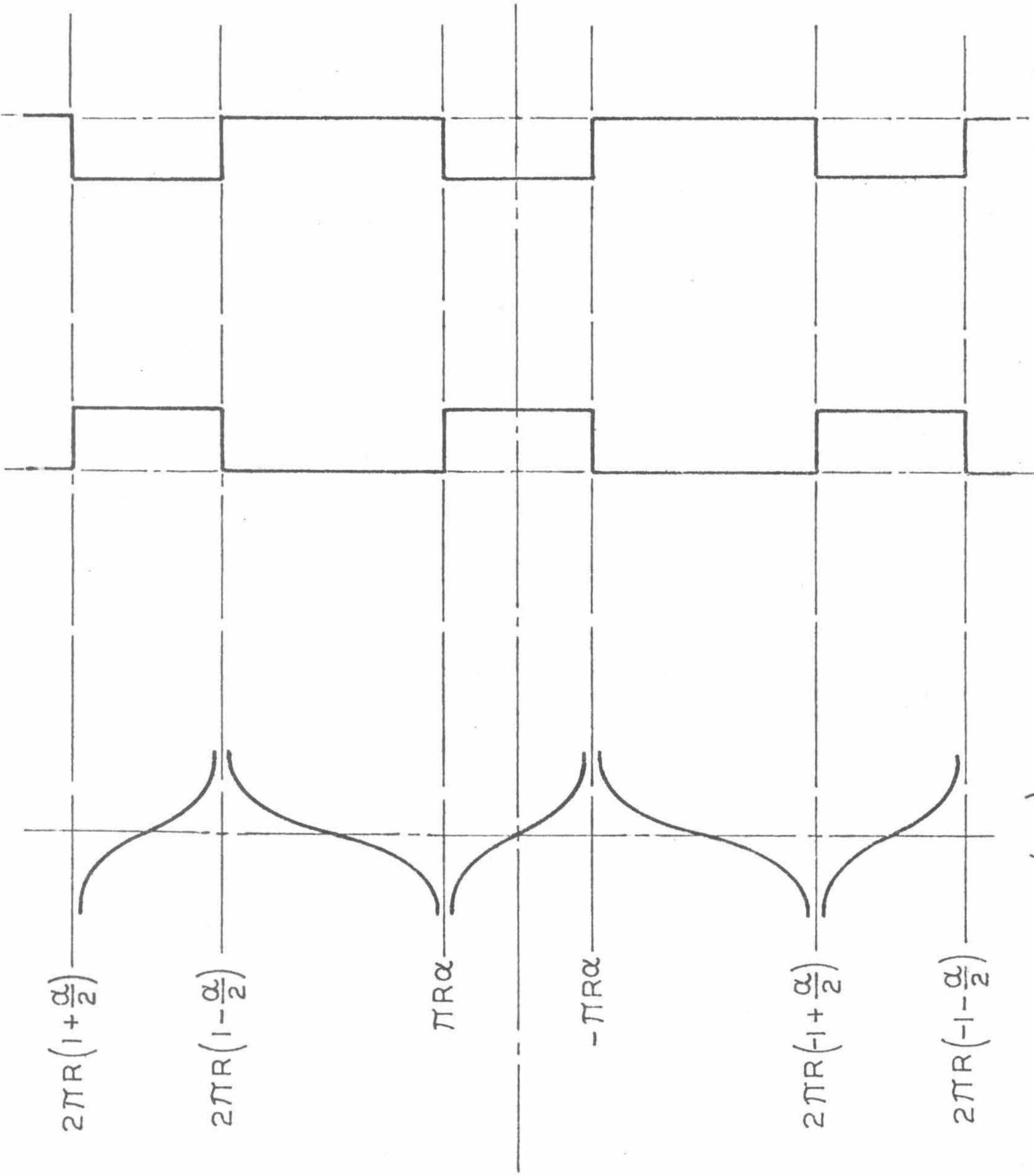

Figure 5. Distribution of real and imaginary parts of $\mathcal{F}(z)$ along y-axis.